

The connection.

By Cartan's formula and formal integrability,
 $d\Theta^\alpha$ cannot have $\Theta^\beta \wedge \Theta^\gamma$ terms \Rightarrow

$$d\Theta^\alpha = \Theta^\beta \wedge \omega_\beta^\alpha + \Theta^i \tau^{\alpha}_i, \quad (1)$$

for some 1-forms $\omega_\beta^\alpha, \tau^{\alpha}_i$. We can (by moving terms of the form $\Theta^i \Theta^\beta$ in the second term to the first), achieve

$$(3) \quad \tau^{\alpha}_i = A_{\bar{\mu}}^{\alpha} \Theta^{\bar{\mu}}, \text{ which determines } \Theta^\beta \wedge \omega_\beta^\alpha.$$

The 1-forms ω_β^α are now of the form

$$\omega_\beta^\alpha = \omega_\beta^\alpha \gamma \Theta^\gamma + \omega_\beta^\alpha \bar{\mu} \Theta^{\bar{\mu}} + \omega_\beta^\alpha \Theta,$$

where $\omega_\beta^\alpha \gamma$ are determined since τ^{α}_i are,

and $\omega_\beta^\alpha \bar{\mu}$ are as well. We may change

$$\omega_\beta^\alpha \rightarrow \omega_\beta^\alpha + b_{\beta\gamma}^\alpha \Theta^\gamma \quad (2)$$

as long as $b_{\beta\gamma}^\alpha \Theta^\beta \wedge \Theta^\gamma = 0$

Conv. We use Lie form $g_{\alpha\bar{\beta}}$ to lower and raise indices, e.g. $A_{\bar{\mu}\bar{\nu}}^\alpha = A_{\bar{\mu}}^\alpha g_{\bar{\nu}\bar{\beta}}$; we conjugate to conjugate indices, $A_{\alpha\beta} = \overline{A_{\bar{\alpha}\bar{\beta}}}$.

Prop (Webster). (1) $A_{\alpha\beta} = A_{\beta\alpha}$

(2) We can choose Φ_{β}^{α} uniquely so that
 $\omega_{\alpha\bar{\beta}} - \omega_{\bar{\beta}\alpha} = dg_{\alpha\bar{\beta}}$.

Pf. Differentiate $d\theta = ig_{\alpha\bar{\beta}}\theta^{\alpha}\wedge\theta^{\bar{\beta}}$ and use (1) \Rightarrow
 $0 = i(dg_{\alpha\bar{\beta}} - \omega_{\alpha}^{\gamma}g_{\gamma\bar{\beta}} - \omega_{\bar{\beta}}^{\bar{\gamma}}g_{\alpha\bar{\gamma}})\wedge\theta^{\alpha}\wedge\theta^{\bar{\beta}}$
 $+ i\theta\wedge(\tau_{\bar{\mu}}\wedge\theta^{\bar{\mu}} + \theta\wedge\tau_{\gamma})$

Substitute (3) $\tau^{\alpha} = A_{\bar{\mu}}^{\alpha}\theta^{\bar{\mu}} \Rightarrow$
 $dg_{\alpha\bar{\beta}} - \omega_{\alpha}^{\gamma}g_{\gamma\bar{\beta}} - \omega_{\bar{\beta}}^{\bar{\gamma}}g_{\alpha\bar{\gamma}} =$
 $A_{\alpha\bar{\beta}\nu}\theta^{\nu} + B_{\alpha\bar{\beta}\bar{\mu}}\theta^{\bar{\mu}},$

where (using $g_{\alpha\bar{\beta}}$ Hermitian)

$A_{\alpha\bar{\beta}\nu} = A_{\nu\bar{\beta}\alpha}$, $B_{\alpha\bar{\beta}\bar{\mu}} = B_{\alpha\bar{\mu}\bar{\beta}}$, $B_{\alpha\bar{\beta}\bar{\mu}} = A_{\bar{\beta}\alpha\bar{\mu}}$

and $\tau_{\bar{\mu}}\wedge\theta^{\bar{\mu}} = 0 (\Leftrightarrow (1))$. Choosing

$\Phi_{\beta}^{\alpha} = A_{\beta}^{\alpha}$ \Rightarrow (2). Details DIY.

□

The Prop determines $\omega_\alpha{}^\beta, \tau^\beta$ uniquely.
 We define a connection on \mathcal{V}

$$\nabla \zeta_\alpha = \omega_\alpha{}^\beta \zeta_\beta,$$

i.e. ∇ is a linear map
 $\xrightarrow[\text{sections of } \mathcal{V}]{} \mathcal{C}^\infty(\mathcal{V}) \rightarrow \mathcal{C}^\infty(\mathcal{V}) \otimes \wedge^1(\mathcal{M})$ \nwarrow 1-forms

that satisfies Leibnitz rule $\nabla(u\zeta) = du \otimes \zeta + u \nabla \zeta$.

Def. ∇ is the Tanaka-Webster connection associated with the contact form θ .

• τ^α is the Tanaka-Webster torsion.

• $(M, \mathcal{V}, \theta, \nabla)$ is a pseudohermitian structure.

• ∇ acts on any tensor bundle on $T^{1,0}, T^{0,1}$ + duals.

• $\omega_\alpha{}^\beta$ are the connection forms, and

$$\begin{cases} d\theta = ig_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}} \\ d\theta^\alpha = \theta^\beta \wedge \omega_\beta{}^\alpha + \theta^\alpha \wedge \tau^\alpha \\ dg_{\alpha\bar{\beta}} = \omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha}, \tau^\alpha = A_{\bar{\mu}}{}^\alpha \theta^{\bar{\mu}}, A_{\alpha\bar{\beta}} = A_{\bar{\beta}\alpha} \end{cases}$$